Approximating finite life-span age-structured population models by means of the survival probability


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Many different numerical methods have been proposed in the literature for the numerical solution of deterministic age-structured population models (see, for example, [1] and the references therein for a general overview). Lately, the more realistic situation in which all individuals of the population have a finite life-span has been considered. However, this situation leads to additional problems in the convergence analysis of the numerical methods.

Our starting point is the Lotka-McKendrick model describing the time evolution of a single and closed population structured by age. Let \( u(a,t) \) represent the age density of individuals at time \( t \). Here \( a \) denotes age, and \( a^\dagger \) is just the maximum age. The vital functions \( \mu(a) \) and \( \beta(a) \) represent, respectively, the age-specific mortality and fertility rates. Thus we consider \( u \) given as solution of the following linear initial-boundary value problem

\[
\frac{\partial u(a,t)}{\partial t} + \frac{\partial u(a,t)}{\partial a} + m(a) u(a,t) = 0, \quad 0 < a < a^\dagger, \quad t > 0,
\]

\[
B(t) = u(0,t) = \int_0^{a^\dagger} \beta(a) u(a,t) \, da, \quad t > 0,
\]

\[
u(a,0) = u_0(a), \quad 0 \leq a < a^\dagger.
\]

The problem consists of a first order partial differential equation, the balance law describing aging and dying, a boundary condition representing the inflow of newborns (\( B(t) \) is the total birthrate at time \( t \)), and an initial condition providing the initial age distribution \( u_0(a) \).

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The survival probability,
\[ \pi(a) = \exp\left\{-\int_0^a m(\sigma) \, d\sigma\right\}, \quad 0 \leq a \leq a^*, \]
that is, the probability at birth of living to age \( a \), is a meaningful quantity in the analysis of this model. In order to assure that the survival probability vanishes at the maximum age, we must assume the following condition about the mortality rate
\[ \int_0^{a^*} m(a) \, da = +\infty, \]
therefore, an unbounded mortality rate must be considered.

In such case, Iannelli and Milner [2] pointed out that standard numerical methods would not work well near \( a^* \). They discuss the numerical approximation of the survival probability, for a general class of unbounded mortality rates, by means of finite-difference methods applied to the numerical solution of a suitable first-order ordinary differential equation which characterizes \( \pi \).

In this work, we propose a different method in order to approximate the survival probability. It is based on the approximation of the integral in the definition of \( \pi \) by means of a quadrature rule, producing a more natural procedure than that presented in [2].

In general, in order to attain a predictable order of convergence, some regularity conditions must be imposed on the integrand function. In this case, we establish the assumptions on the mortality function near the maximum age required to achieve an optimal rate of convergence in representative situations. In this respect, the analysis we carry out in this work concerns mortality rates which are significant for the real applications of the problem, including, for example, the test function used in [2].

Besides that, we explore, analytically and from an experimental point of view, the value of this numerical procedure in order to approximate the survival probability in some representative cases (as, for example, in [2] and [3]).

Finally, we present a useful technique in order to apply the approximation of the survival probability in the numerical solution of the previous age structured population model.

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References

