# Semilocal convergence of an efficient fifth order method under weaker conditions

S. Singh [1] Department of Mathematics, N.I.T. Hamirpur
E. Martínez [2] , Instituto de Matemática Pura y Aplicada, UPV
D.K. Gupta [3] and A. Kumar [4] Department of Mathematics, I.I.T. Kharagpur

**Abstract**

The semilocal convergence of an efficient fifth order iterative method is established under weaker conditions for solving nonlinear equations. It is done by assuming omega continuity condition on second order Fréchet derivative. The novelty of our work lies in the fact that several examples are available where Lipschitz and Hölder condition fails but omega condition holds. Existence and uniqueness theorem is established along with R-order and error bounds. The R-order is found to be $4 + q$, $q \in (0, 1]$. Numerical experiments involving nonlinear integral equations are performed to show the applicability of the method. Finally the existence and uniqueness balls are obtained along with error bounds for all the examples.

*Keywords:* Semilocal convergence; Lipschitz condition; Hölder condition; Hammerstein integral equation and Dynamical Systems
**Mathematical Subject Classification** 65H10, 47H99

## 1. Introduction

Let $X$ and $Y$ are Banach spaces and consider solving

$$G(x) = 0$$

where $G : \Omega \subseteq X \to Y$ be a nonlinear operator in an open convex domain $\Omega_0 \subseteq \Omega$. Solution of various real life problems such as dynamical systems, boundary value problems etc. are obtained by solving these equations (see, [1, 2]). The most well known quadratically convergent Newton's method to solve (1.1) is defined for $k \geq 0$, by

$$x_{k+1} = x_k - \Gamma_k G(x_k) \tag{1.1}$$

where, $\Gamma_k = G^{'}(x_k)^{-1}$ and $x_0$ is the initial point. Various modification of Newton's method are proposed to increase the order of convergence and efficiency. In literature [3, 4, 5, 6, 7, 8], authors have established the semilocal convergence of higher order iterative methods under various continuity conditions.

[1]Corresponding* sukhjit@maths.iitkgp.ernet.in
[2]eumarti@mat.upv.es
[3]dkg@maths.iitkgp.ernet.in
[4]abhimanyukumar01991@gmail.com

Recently, the semilocal convergence of an efficient fifth order method is established in [9] under Lipschitz condition on $F^{''}$. It is given for $k = 0, 1, 2 \dots$ by

$$\begin{aligned} y_k &= x_k - \Gamma_k G(x_k), \\ z_k &= y_k - \Gamma_k G(y_k), \\ x_{k+1} &= z_k - G^{'}(y_k)^{-1} G(z_k), \end{aligned} \tag{1.2}$$

In real life applications, various numerical examples are available which neither satisfies Lipschitz nor Hölder condition. This motivate us to establish the semilocal convergence of an efficient fifth order method under weaker conditions.

## 2. Semilocal convergence analysis

Let $\Gamma_0 = G^{'}(x_0)^{-1} \in BL(Y, X)$ exists at $x_0 \in \Omega$, where $BL(Y, X)$ denotes the set of bounded linear operators from $Y$ to $X$ and the following conditions hold.
(1) $\|\Gamma_0\| \leq \beta_0$
(2) $\|\Gamma_0 G(x_0)\| \leq \eta_0$
(3) $\|G^{''}(x)\| \leq M$
(4) $\|G^{''}(x) - G^{''}(y)\| \leq \omega(\|x - y\|), x, y \in \Omega$, for a continuous non-decreasing real function $\omega(x)$, $x > 0$, $\omega(0) \geq 0$ such that, $\omega(tx) \leq t^q \omega(x)$ for $t \in [0, 1]$, $x \in (0, \infty)$ and $q \in [0, 1]$.
Let $r_0 = M\beta_0\eta_0$, $s_0 = \beta_0\eta_0\omega(\eta_0)$ and define sequences $\{r_k\}$, $\{s_k\}$ and $\{\eta_k\}$ for $k = 0, 1, 2 \dots$, by

$$r_{k+1} = r_k \phi(r_k)^2 \psi(r_k, s_k), \tag{2.1}$$

$$s_{k+1} = s_k \phi(r_k)^{2+q} \psi(r_k, s_k)^{1+q}, \tag{2.2}$$

$$\eta_{k+1} = \eta_k \phi(r_k) \psi(r_k, s_k), \tag{2.3}$$

where,

$$\phi(u) = \frac{1}{1 - ug(u)}, \tag{2.4}$$

$$g(u) = \left(1 + \frac{u}{2} + \frac{u^2}{2(1-u)}\left(1 + \frac{u}{4}\right)\right), \tag{2.5}$$

and

$$\begin{aligned} \psi(u, v) \quad &= \quad \frac{u^2}{2(1-u)}\left(1 + \frac{u}{4}\right)\left[\frac{v}{1+q}\left(\frac{u^{1+q}}{2^{1+q}} + \frac{1}{2+q}\left(\frac{u^2}{2(1-u)}\left(1 + \frac{u}{4}\right)\right)^{1+q}\right)\right. \\ &+ \quad \left.\frac{u}{2}\left(u + \frac{u^2}{2(1-u)}\left(1 + \frac{u}{4}\right)\right)\right]. \end{aligned} \tag{2.6}$$

Let $h(u) = g(u)\,u - 1$. Since, $h(0) = -1$ and $g(u)$ is increasing function, therefore, $h(u)$ has a real root $\nu$. If $u \in (0, \nu)$, we get $g(u)\,u < 1$.

**Lemma 2.1.** *Let $\phi(u)$, $g(u)$ and $\psi(u, v)$ are given by (2.4), (2.5) and (2.6) respectively. If $0 < r_0 < \nu$ and $\phi(r_0)^2\psi(r_0, s_0) < 1$, then*
*(i) $\phi(u)$ and $\psi(u)$ are increasing functions and $\phi(u) > 1$, $g(u) > 1$ for $u \in (0, \nu)$.*
*(ii) $\psi(u, v)$ is an increasing function of $u$, for $u \in (0, \nu)$.*
*(iii) $\{r_k\}, \{s_k\}$ and $\{\eta_k\}$ are decreasing sequences and $r_k g(r_k) < 1$ as well as $\phi(r_k)^2\psi(r_k, s_k) < 1$ for $k \geq 0$.*

**Lemma 2.2.** *Let $\phi(u)$ and $\psi(u,v)$ are given by (2.4) and (2.6), respectively. If $\gamma \in (0,1)$ then $\phi(\gamma t) < \gamma\phi(t)$ and $\psi(\gamma u, \gamma^{1+q} v) < \gamma^{3+q}\psi(u,v)$.*

**Lemma 2.3.** *Let $\gamma = \phi(r_0)^2\psi(r_0,s_0)$, $0 < r_0 < \nu$ and $\delta = \frac{1}{\phi(r_0)}$. Then,*
*(i) $r_k \leq \gamma^{(4+q)^{k-1}} r_{k-1} \leq \gamma^{\frac{(4+q)^k - 1}{3+q}} r_0$ and $s_k \leq \left(\gamma^{(4+q)^{k-1}}\right)^{1+q} s_{k-1} \leq \left(\gamma^{\frac{(4+q)^k-1}{3+q}}\right)^{1+q} s_0$.*
*(ii) $\phi(r_k)\psi(r_k,s_k) \leq \dfrac{\gamma^{(4+q)^k}}{\phi(r_0)}$ $\forall\ k \in N$.*
*(iii) $\eta_k \leq \gamma^{\frac{(4+q)^k-1}{3+q}} \delta^k \eta_0$.*

Using the above results, we will establish the following recurrence relations and convergence theorem.
(I) $\|\Gamma_k\| \leq \phi(r_{k-1})\|\Gamma_{k-1}\|$,
(II) $\|\Gamma_k G(x_k)\| \leq \phi(r_{k-1})\psi(r_{k-1}, s_{k-1})\eta_{k-1}$,
(III) $M\|\Gamma_k\|\|\Gamma_k G(x_k)\| \leq r_k$,
(IV) $\|\Gamma_k\|\|\Gamma_k G(x_k)\|\omega\left(\|\Gamma_k G(x_k)\|\right) \leq s_k$,
(V) $\|x_k - x_{k-1}\| \leq g(r_{k-1})\eta_{k-1}$.

**Theorem 2.1.** *Let $r_0 = M\beta_0\eta_0 < \nu$, $s_0 = \beta_0\eta_0\omega(\eta_0)$ and assumptions (1)-(4) hold. Then for $\overline{B}(x_0, R\eta_0) \subseteq \Omega$, where $R = \dfrac{g(r_0)}{1-\delta\gamma}$, the sequence $\{x_k\}$ generated by (1.2) converges to the solution of (1.1). Moreover, $y_k, z_k, x_{k+1}, x^* \in \overline{B}(x_0, R\eta_0)$ and $x^*$ is the unique solution in $B\left(x_0, \frac{2}{L_1\beta_0} - R\eta_0\right) \cap \Omega$. The error bound for iterates is given as follows.*

$$\|x_k - x^*\| \leq g(r_0)\delta^k \frac{\gamma^{\frac{(4+q)^k-1}{3+q}}}{1-\delta\gamma^{(4+q)^k}}\eta_0.$$

## 3. Numerical examples

In this section, we consider nonlinear Hammerstein type integral equation which arises in dynamical model of a chemical reactor (see, [10]) given by

$$x(r) + \sum_{i=1}^{m} \int_a^b K_i(r,s)S_i(x(s))ds = f(r),\ r \in [a,b], \tag{3.1}$$

where functions $f$, $K_i$ and $S_i$ for $i = 1, 2, \dots m$ are known, the solution $x$ is to be determined and $-\infty < a < b < +\infty$. In order to solve (3.1), we have to solve

$$G(x)(u) = x(u) + \sum_{i=1}^{m} \int_a^b K_i(u,v)S_i(x(v))dv - f(u) \tag{3.2}$$

If $S_i^{'}(x(u))$ is $(M_i, \alpha_i)$- Hölder continuous in $\Omega$, then, under max-norm, we have

$$\|G^{''}(x) - G^{''}(y)\| \leq \sum_{i=1}^{m} M_i\|x-y\|^{\alpha_i},\ M_i \geq 0,\ \alpha_i \in [0,1],\ \forall\ x,y \in \Omega. \tag{3.3}$$

For different $\alpha_i$, $G^{''}$ neither satisfies Lipschitz nor Hölder condition but satisfies the weaker $\omega$-condition. Using the proposed study, we obtain the existence and uniqueness balls for the solution along with error bounds.

## 4. Conclusions

Using recurrence relations, semilocal convergence of an efficient fifth order iterative method is presented under weaker conditions for solving nonlinear equations. The convergence theorem is established along with error bounds. Different examples involving nonlinear integral equations are solved to show the applicability of the approach. Existence and uniqueness balls are obtained along with error bounds for the considered examples.

## References


[1] Argyros, I.K., Hilout, S., Tabatabai, M.A.: Mathematical Modelling with Applications in Biosciences and Engineering. Nova Publishers, New York, (2011)

[2] Argyros, I.K., Hilout, S.: Numerical methods in Nonlinear Analysis. World Scientific Publ. Comp. New Jersey (2013)

[3] Kantorovich, L.V., Akilov, G.P.: Functional Analysis, Pergamon Press, Oxford, 1982

[4] Amat, S., Hernández, M.A., Romero, N.: A modified Chebyshev's iterative method with at least sixth order of convergence. Appl. Math. Comput. 206, 164-174 (2008)

[5] Hernández, M.A.: Chebyshev's approximation algorithms and applications. Computers Math. Applic. 41, 433-445 (2001)

[6] Hueso, J.L., Martínez, E.: Semilocal convergence of a family of iterative methods in Banach spaces. Numer. Algorithms. 67, 365-384 (2014)

[7] Wang, X., Kou,J., Gu, C.: Semilocal convergence of a class of Modified Super-Halley Methods in Banach Spaces. J. Optim. Theory Appl. 153, 779-793 (2012)

[8] Wu, Q., Zhao, Y.: Newton-Kantorovich type convergence theorem for a family of new deformed Chebyshev method. Appl. Math. Comput. 192, 405-412 (2008)

[9] Singh, S., Gupta, D.K., Martínez, E., Hueso, J.L. : Semilocal convergence analysis of an iteration of order five using recurrence relations in Banach spaces. Mediterr. J. Math. 13, 4219-4235 (2016)

[10] Bruns, D., Bailey, J., : Nonlinear feedback control for operating a nonisothermal cstr near an unstable steady state. Chemical Engineering Science 32, 257-264 (1977).