Stability of parametric family of iterative methods for root-finding *

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Nonlinear equations $F(x) = 0$, where $F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear function defined in a convex set D, are often used for modeling real problems arising in science and engineering as, for example, in the analysis of dynamical models of chemical reactors, preliminary orbit determination of satellites, in radioactive transfer, in economics modeling problems, transport theory etc. These problems lead to a rich blend of mathematics, numerical analysis and computing science.

In general, for solving these equations iterative methods must be used. The proliferation of iterative schemes for solving nonlinear equations ($n = 1$) has been spectacular in the last years (we can see an overview in [1]). The most of them are variants of Newton's method obtained by means of different procedures. The direct composition of known methods with a later treatment to reduce the number of functional evaluations, the weight function procedure, etc are some of the most used techniques for designing new schemes.

In the literature, iterative methods are analyzed under different points of view. A research area that is getting strength nowadays consists of applying discrete dynamics techniques to the associated fixed point operator of iterative methods. The dynamical behavior of such operators when applied on the simplest function (a low degree polynomial) gives us relevant information about its stability and performance. This study is focused on the asymptotic behavior of fixed points, as well as in its associated basins of attraction. Indeed, in case of families of iterative schemes, the analysis of critical points (where the derivative of the rational function is null), different from the roots of the polynomial, not only allows to select those members of the class with better properties of stability, but also to classify iterative methods of the same order in terms of their dynamics.

In the last years, the use of tools from complex dynamics has allowed the researchers in this area of numerical analysis to deep in the understanding of the stability of iterative schemes (see, for example, [2–6]). The analysis, in these terms, of the rational function R associated to the iterative procedure applied on quadratic polynomials, gives us valuable information about its role

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on the convergence's dependence on initial estimations, the size and shape of convergence regions and even on a possible convergence to fixed points that are not solution of the problems to be solved or to attracting cycles. Moreover, if a family of parametric schemes is studied, the most stable elements of the class can be chosen, by means of an appropriate use of the parameter plane.

In this paper, we present a parametric family of two steps iterative methods whose iterative expression is

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \left(\alpha_1 + \alpha_2 \frac{f'(y_k)}{f'(x_k)} + \alpha_3 \left(\frac{f'(y_k)}{f'(x_k)}\right)^2\right) \frac{f(y_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{1}$$

where α_1, α_2 and α_3 are parameters.

The following result establishes the convergence of iterative method (1).

Theorem 1 *Let $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be sufficiently differentiable at each point of an open interval D such that $\bar{x} \in D$ is a simple solution of equation $f(x) = 0$ and the initial estimation x_0 is close enough to $\bar{x}$. Then, sequence $\{x_k\}_{k\geq 0}$ obtained from expression (1) converges to $\bar{x}$ with order 4 when $\alpha_2 = 3 - 2\alpha_1$ and $\alpha_3 = -2 + \alpha_1$, being in this case the error equation*

$$e_{k+1} = (13 - 4\alpha_1) C_2^3 e_k^4 + O(e_k^5)$$

where $C_j = \dfrac{1}{j!} \dfrac{f^{(j)}(\bar{x})}{f'(\bar{x})}$, $j = 2, 3, \ldots$

There exists an element of the family, corresponding to $\alpha_1 = 13/4$, with fifth-order of convergence.

In order to analyze the dynamical behavior of family (1) on quadratic polynomials we choose a generic one $p(z) = (z - a)(z - b)$. If we apply (1) on $p(z)$, a rational operator depending to parameters a, b and α_1, $T_{p,\alpha_1,a,b}(z)$, is obtained.

By means of the conjugacy map $h(z) = \dfrac{z-a}{z-b}$, (a Möbius transformation), with the following properties:

$$\text{i)}\ \ h(\infty) = 1, \qquad \text{ii)}\ \ h(a) = 0, \qquad \text{iii)}\ \ h(b) = \infty,$$

operator $T_{p,\alpha_1,a,b}(z)$ on quadratic polynomials is conjugated to operator $O_{\alpha_1}(z)$,

$$O_{\alpha_1}(z) = \left(h \circ T_{p,\alpha_1,a,b} \circ h^{-1}\right)(z) = -z^4 \frac{13 - 4\alpha_1 + 14z + 14z^2 + 6z^3 + x^4}{-1 - 6z - 14z^2 - 14z^3 - 13z^4 + 4\alpha_1 z^4}. \tag{2}$$

We analyze the fixed and critical points of operator $O_{\alpha_1}(z)$. Some results about the stability of the fixed point are obtained and the behavior of the independent free critical points, used as initial guess, give us interesting parameter planes. From these parameter planes we can extract valuable information about the stability of the different members of the family. Stable and pathological behaviors are obtained depending on parameter. We choose values of α_1 which give us stable iterative schemes and other ones with chaotic numerical deportment. The dynamical planes of all these cases are studied.

We check the numerical behavior of both types of methods (stable and unstable) on a collection of academical test functions.

Two important generalizations can be made with family (1). Firstly, Sharma and Arora in [7], have extended this class of iterative methods for solving nonlinear systems $F(x) = 0$ in a natural way

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - \Big(\alpha_1 + \alpha_2 [F'(x^{(k)})]^{-1} F'(y^{(k)}) + \alpha_3 \left([F'(x^{(k)})]^{-1} F'(y^{(k)})\right)^2\Big) [F'(x^{(k)})]^{-1} F(y^{(k)}), \end{aligned} \quad (3)$$

where α_1, α_2 and α_3 are parameters.

A similar result as Theorem 1 was established in [7] for the class (3) in the case of $\alpha_1 = 13/4$. On the other hand, by adding a new step in the iterative expressions (1) or (3), with the same structure as the second step we can obtain a parametric family of order eight. This idea can be extended for obtaining a parametric family of arbitrary order for solving nonlinear equations or nonlinear systems.

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