A discontinuous Galerking framework for option pricing problems with stochastic volatilities

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Abstract

The modern theory of option pricing was based on the ideas of Black-Scholes framework firstly published in [2]. Nowadays it is accepted that these models (see [14] for a complete overview) are not sufficiently accurate in capturing the real world features of the stock markets, because its idealized assumptions do rarely hold in practice. One of them is the limitation of constant volatility, which can be relaxed by models with assumptions of stochastic volatility. There are a number of models, e.g. models by Hull-White [8], Scott [10], Stein-Stein [12] or Heston [5]. A standard mathematical approach to these models leads to partial differential equations (PDEs) completed by the system of boundary and terminal (initial) conditions, where the (semi) closed-form solution is not always attainable and one has to construct numerical approximations.

The performance demands on the numerical valuation process is very high and several techniques have been developed to obtain efficient pricing algorithms over last years, from lattice/trees methods [3], over finite difference schemes [11] to finite element approaches [1, 13]. These methods have also its limitations in the treatment of numerical option pricing under more complex market conditions such as an assumption of a stochastic volatility. Therefore, it should be convenient to follow novel alternative option pricing schemes which are also robust with respect to different market conditions that need to be taken into account.

In this work we propose a numerical technique based on the discontinuous Galerkin method (see [9]) to unify the option pricing under a wide spectrum of volatility models with one stochastic process. Let us consider a financial asset whose price is given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t, \tag{1}$$

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where $\mu S_t dt$ is a drift term with a constant rate μ, W_t is a Brownian motion and σ_t is the volatility. Further, we assume that $\sigma_t = f(y_t)$ for some positive function f and y_t is the general driving process

$$dy_t = A(y_t, t)dt + B(y_t, t)dZ_t, \tag{2}$$

where nothing will be assumed about the functions $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$, and the second Brownian motion Z_t is correlated to W_t as $dW_t dZ_t = \rho dt$ with factor $\rho \in (-1; 1)$.

Note that a suitable choice of A and B can include different processes such as lognormal, mean-reverting OU process or CIR process. Secondly, the setting of f together with the value of the correlation coefficient leads to particular stochastic volatility models, the frequently used models are listed in the following table (α, β and m are positive constants).

Model	y_t process		function f	correlation
Heston	$A(y,t) = \alpha(m-y)$	$B(y,t) = \beta\sqrt{y}$	$f(y) = \sqrt{y}$	$\rho \neq 0$
Hull-White	$A(y,t) = \alpha y$	$B(y,t) = \beta y$	$f(y) = \sqrt{y}$	$\rho = 0$
Scott	$A(y,t) = \alpha(m-y)$	$B(y,t) = \beta$	$f(y) = \exp(y)$	$\rho = 0$
Stein-Stein	$A(y,t) = \alpha(m-y)$	$B(y,t) = \beta$	$f(y) = \lvert y \rvert$	$\rho = 0$

Next, we consider a European option on the financial asset S (for the sake of clarity we omit the subscript t) with maturity T and assume instantaneous risk-free interest rate r. The option price $V(S, y, t)$ depends on the underlying asset S, the driving process y and the actual time t. A common approach based on no arbitrage principle, Ito's stochastic calculus and a construction of a sophisticated portfolio leads to the pricing PDE, which can be decomposed in the following way

$$\frac{\partial V}{\partial t} + \mathcal{L}^f_{BS}(V) + \mathcal{L}^f_{corr}(V) + \mathcal{L}_{proc}(V) - \mathcal{L}_{prem}(V) = 0, \quad S > 0, y \in \mathbb{R}, t \in [0, T), \tag{3}$$

where differential operators (representing Black-Scholes part, correlation, driving process and premium) are defined as

$$\begin{aligned}
\mathcal{L}^f_{BS}(V) &= \frac{1}{2} f^2(y) S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV, \\
\mathcal{L}^f_{corr}(V) &= \rho B(y,t) S f(y) \frac{\partial^2 V}{\partial S \partial y}, \\
\mathcal{L}_{proc}(V) &= \frac{1}{2} B^2(y,t) \frac{\partial^2 V}{\partial y^2} + A(y,t)\frac{\partial V}{\partial y} \\
\mathcal{L}_{prem}(V) &= B(y,t)\Lambda(S,y,t)\frac{\partial V}{\partial y}.
\end{aligned}$$

Let us mention that the term $\mathcal{L}_{prem}(V)$ is called a premium on the volatility risk, where Λ is usually defined as

$$\Lambda(S, y, t) = \rho\frac{\mu - r}{f(y)} + \sqrt{1-\rho^2}\gamma(S, y, t), \tag{4}$$

where the function γ is the risk premium factor (market price of the volatility risk) and can be chosen arbitrarily.

In order to obtain the initial boundary value problem, the pricing equation has to be restricted on a bounded domain $\Omega \subset I\!R^+ \times I\!R$ and closed with the set of initial and boundary conditions. Since (3) is backward in time, the initial (terminal) condition at maturity date T is given by the payoff function $V^0(S)$ depending on the type of a option (call or put). Due to the localization of (3) on Ω, one has to prescribe mixed boundary conditions on appropriate parts of $\partial\Omega$, which are chosen compatible with the payoff and using knowledge on the asymptotic behavior of options.

Since the pricing equation is closely related to the convection-diffusion equation, which exhibits parabolic and hyperbolic behavior in dependency on a proportion of the convection and diffusion parts, the numerical schemes for solving of such equation should be constructed with respect to these properties. Here, we extend the DG framework from [6, 7] with some modifications with respect to the unified approach to studied volatility models.

We construct solution $V_h = V_h(t)$ from the finite dimensional space S_h^p consisting from piecewise polynomial, generally discontinuous, functions of the p-th order defined on the domain Ω. Using a method of lines leads to a system of the ordinary differential equations

$$\frac{d}{dt}(V_h, \varphi_h) + \mathcal{A}_h(V_h, \varphi_h) = 0 \quad \forall\, \varphi_h \in S_h^p,\ \forall\, t \in (0, T) \tag{5}$$

where the initial condition is given by V^0, $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ and the form $\mathcal{A}_h(\cdot,\cdot)$ stands for the DG semi-discrete formulation of the operator $\mathcal{L}_{BS}^f + \mathcal{L}_{corr}^f + \mathcal{L}_{proc} - \mathcal{L}_{prem}$. Consequently, we realize the discretization in time by Crank-Nicolson scheme and obtain at each time level $t_m \in [0, T]$ the sparse matrix equation

$$\left(\mathbf{M} + \frac{\tau}{2}\mathbf{A}\right) V_{m+1} = \left(\mathbf{M} - \frac{\tau}{2}\mathbf{A}\right) V_m + \frac{\tau}{2}\left(F_{m+1} + F_m\right), \tag{6}$$

where the vector V_m is related to the DG solution $V_h(t_m)$, the matrix $\mathbf{M}$ to the mass matrix, the matrix $\mathbf{A}$ to the bilinear form $\mathcal{A}_h$ and the vector F_m to the boundary conditions, respectively. The existence and uniqueness of the DG solution is guaranteed by the invertibility of the corresponding system matrix in (6).

Finally, the numerical results are presented on artificial benchmarks as well as on reference market data in order to illustrate the usage of DG method to pricing of options under such stochastic volatility models. All computations are carried out with an algorithm implemented in the solver Freefem++, the detailed description can be found in [4].

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References

[1] Y. Achdou, and O. Pironneau, *Computational Methods for Option Pricing.* Frontiers in Applied Mathematics. SIAM, Philadelphia, 2005.

[2] F. Black, and M. Scholes, The pricing of options and corporate liabilities. *Journal of Political Economy*, vol. **81**, 1973, pp. 637-659.

[3] J.C. Cox, S. Ross, and M. Rubinstein, Option pricing: A simplified approach. *Journal of Financial Economics*, vol. **7**, 1979, pp. 229-264.

[4] F. Hecht, New development in FreeFem++. *Journal of Numerical Mathematics*, vol. **20**, no. 3-4, 2012, pp. 251-265.

[5] S. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, vol. **6**, 1993, pp. 327-343.

[6] J. Hozman, and T. Tichý, A discontinuous Galerkin method for numerical pricing of European options under Heston stochastic volatility. AIP Conference Proceedings, vol. **1789**, article no. 030003, 2016, doi: 10.1063/1.4968449.

[7] J. Hozman, and T. Tichý, DG method for numerical pricing of multi-asset Asian options—The case of options with floating strike, *Applications of Mathematics*, vol. **62**, no. 2, 2017, pp. 1-25.

[8] J. Hull, and A. White, The Pricing of Options on Assets with Stochastic Volatilities. *Journal of Finance*, vol. **42**, 1987, pp. 281-300.

[9] B. Riviére, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation.* Frontiers in Applied Mathematics. SIAM, Philadelphia, 2008.

[10] L.O. Scott, Option Pricing When the Variance Changes Randomly: Theory, Estimation, and an Application. *Journal of Financial and Quantitative Analysis*, vol. **22**, 1987, pp. 419-438.

[11] R.U. Seydel, *Tools for Computational Finance.* Springer, Berlin, fourth edition, 2008.

[12] E.M. Stein, and J.C. Stein, Stock Price Distributions with Stochastic Volatility: An Analytic Approach. *Journal of Financial and Quantitative Analysis*, vol. **4**, 1991, pp. 727-752.

[13] J. Topper, *Financial Engineering with Finite Elements.* Wiley, 2005.

[14] P. Willmott, J. Dewynne, and J. Howison, *Option pricing: mathematical models and computations.* Oxford financial press. 1993.