Iterative methods based on HSS and GPSS schemes to solve systems of nonlinear equations.

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The construction of iterative methods for approximating the solution of systems of nonlinear equations, $G(\mathbf{x}) = A\mathbf{x} - \varphi(\mathbf{x}) = 0$, where $A \in \mathbb{C}^{n\times n}$ is a large, sparse, positive definite matrix and $\varphi : \mathbb{D} \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a continuously differentiable function, is an important and interesting task in numerical analysis and applied scientific branches. With the improvement of computers, the problem of solving large scale nonlinear systems by numerical methods has gained importance.

The most common root-finding method for systems of nonlinear equations is the classical Newton method as

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - [G'(\mathbf{x}^{(k)})]^{-1}G(\mathbf{x}^{(k)}),\ k = 0, 1, 2, \ldots.$$

But the second order Newton method and many higher order schemes that in recent decades have been introduced need to compute or approximate Jacobian matrix in the obtained points at each step, which is a very time-consuming process. Therefore, introducing any scheme which does not need Jacobian matrix is welcome. The development of Jacobian free softwares to solve problems which are formulated by nonlinear partial differential equations is of increasing interest to simulate practical engineering processes. In fact, Jacobian free methods for solving nonlinear systems are very important and form an attractive area of research.

When the problem is weakly nonlinear, that is, when the linear term $A\mathbf{x}$ is strongly dominant over the nonlinear term $\varphi(\mathbf{x})$, Zhu [1] used the Picard iteration method $A\mathbf{x}^{(\mathbf{k+1})} - \varphi(\mathbf{x}^{(\mathbf{k})})$ to solve these systems of weakly nonlinear equations. Therefore any iteration of Picard method contains two kinds of iterations. The linear iteration to solve a linear system such as $A\mathbf{x} = \mathbf{b}$ which is called the inner iteration, whereas the nonlinear iteration to compute $\mathbf{x}^{(k)}$ is called the outer iteration. In this paper we use respectively, superscript k and subscript l to show outer and inner iterations of Picard method. Some of the most famous inner iteration methods are Jacobi, Gauss-Siedel, successive overrelaxation (SOR), accelerated overrelaxation (AOR) and Krylov subspace methods. They are based on splitting matrix A as $A = M - N$. When, e.g., Krylov subspace method is used as the inner iteration, we are using an inexact kind of Picard iteration method for solving the weakly nonlinear problem.

Zhu further in [1] propose an asymmetric Hermitian and skew-Hermitian splitting (AHSS) to solve the corresponding linear systems in each iteration. This naturally leads to an inexact Picard

iteration method, called Picard-AHSS iteration method. AHSS algorithm to solve the linear system $A\mathbf{x} = \mathbf{b}$ is described as follows:

Suppose that H and S be Hermitian and skew-Hermitian parts of A, respectively, that is, $A = H + S$ with $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$. Now, for an initial guess $\mathbf{x}_0 \in \mathbb{C}^n$, and positive constants α and tol, in AHSS scheme, one computes $\mathbf{x}_l$ for $l = 1, 2, \cdots$ by

$$\begin{cases} (\alpha I + H)\mathbf{x}_{l+\frac{1}{2}} = (\beta I - S)\mathbf{x}_l + \mathbf{b} \\ (\beta I + S)\mathbf{x}_{l+1} = (\alpha I - H)\mathbf{x}_{l+\frac{1}{2}} + \mathbf{b} \end{cases} \tag{1}$$

where α is a given positive constant and I denotes the identity matrix. Stopping criteria for relations (1) is $\|\mathbf{b} - A\mathbf{x}_l\| \leq \text{tol}\,\|\mathbf{b} - A\mathbf{x}_0\|$, for an initial guess $\mathbf{x}_0$ and a given tolerance tol.

When the linear term is not strongly dominant over nonlinear term, usually Picard iteration numerically is not successful to solve nonlinear equations. As we mentioned above, using Newton's or Newton-like iterations are so expensive. Jacobian-free Newton-Krylov (JFNK) methods [2–4] are efficient schemes to solve large scale nonlinear equations. One of the most important advantages of JFNK methods is that there is no need to form and store the Jacobian matrix of the nonlinear system when JFNK is used. In fact, JFNK is an inexact Newton's method in which the inversion of the Jacobian matrix is performed to arbitrary precision using a Krylov-method (usually GMRES), and the Jacobian matrix is never computed. Each iteration of JFNK needs a nested "inner" iteration and the bulk of the computational effort is expended in the "inner" Krylov inversion of the Jacobian at each "outer" Newton step. At the end of each iteration, the accumulated Krylov space is expected to change minimally during the final Newton steps when a similar space will be rebuilt in the next Newton iteration. Consequently, at the end of each iteration, JFNK removes information that may be of use in successive iterations.
In this paper we change Picard iteration to obtain a scheme that in some cases performs better than ordinary Picard iteration. We present two new Jacobian free methods to solve weakly nonlinear and nonlinear systems. In AHSS scheme when $\alpha = \beta$ the method is called HSS and it is unconditionally convergent. HSS is used to solve the inner iterations in these two new methods. The new algorithms which we present in this paper combine aspects from many of the previous approaches. However, they are distinguished from all of the aforementioned approaches because they possess all of the following desirable characteristics:

(i) They are fully matrix-free.

(ii) They are simple to use.

(iii) They not only solve weakly nonlinear systems but also can perform better than ordinary Picard-Iteration to solve nonlinear systems.

(iv) They are good PDE solvers.

We check the numerical behavior of our method on a two-dimensional nonlinear convection-diffusion equation

$$\begin{cases} -(u_{xx} + u_{yy}) + q(u_x + u_y) = -e^u - \sin(1 + u_x + u_y), & (x, y) \in \Omega \\ u(x, y) = 0, & (x, y) \in \partial\Omega \end{cases} \tag{2}$$

where $\Omega = (0,1) \times (0,1)$, $\partial\Omega$ is its boundary and q is a positive constant for measuring magnitude of the convection term. Applying a five-point finite difference scheme to the diffusive term and the central difference scheme to the convective term, yields a system of nonlinear equations as

$$A\mathbf{u} = h^2\psi(\mathbf{u}) \tag{3}$$

We compare the numerical results given by our method and the corresponding results by Picard iteration and Nonlinerar Picard iteration proposed in [5], obtaining good results.

References

[1] M.-Z. Zhu, On asymmetric hermitian and skew-hermitian splitting iteration methods for weakly nonlinear systems, J Comput. Anal. Appl. 14(1) (2012) 1521-1398.

[2] P.N. Brown, Y. Saad, Hybrid Krylov methods for nonlinear systems of equations, SIAM J. Stat. Comput. 11(1990) 450–481.

[3] P.N. Brown, Y. Saad, Convergence theory of nonlinear Newton-Krylov algorithms, SIAM J. Optim. 4(1994) 297–330.

[4] D.A. Knoll, D.A. Keyes, Jacobian-free Newton-Krylov methods: a survey of approaches and applications, J. Comput. Phys. 193 (2004) 357–397.

[5] M.Z. Zhu, Modified iteration methods based on the asymmetric HSS for weakly nonlinear systems, J. Comput. Anal. Appl. 15(1)(2013) 188–195.