Iterative Solution of Rank Deficient Least Squares Problems

J. Cerdán†, **D. Guerrero***, J. Marín † and J. Mas†

Abstract

Linear least-squares (LS) problems arise in many large-scale applications of the science and engineering as neural networks, linear programming, exploration seismology, image processing,... The LS problem is formulated as follows. Solve

$$\min_x \|b - Ax\|_2, \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$ $(m \geq n)$ is large and sparse and $b \in \mathbb{R}^m$. This problem can be solved iteratively using the Conjugate Gradient for Least Squares (CGLS) method [1], that implicitly applies the conjugate gradient method to the normal equations

$$A^T Ax = A^T b. \tag{2}$$

The successful application of an iterative method often needs a good preconditioner in order to achieve fast convergence rates. When the matrix A has full rank, the preferred method of choice is computing an incomplete Cholesky factorization (IC) of the symmetric and positive definite matrix $C = A^T A$, see [2]. If the matrix A is rank deficient then, the matrix C is a semidefinite positive matrix and the Cholesky factorization suffers breakdown because negative or zero pivots are encountered. Thus, rank deficient LS problems are in general much more harder to solve. In this work we study the solution of this kind of systems by preconditioned iterations.

The idea is to compute an IC factorization of the shifted matrix

$$C_\alpha = A^T A + \alpha I. \tag{3}$$

The shift α is referred to as a Tikhonov regularization parameter. Tipically α is chosen such that the IC factorization is breakdown free, and at the same time it should be chosen as small as possible in order to get a preconditioner

*Departamento de Ciencias Matemáticas, Universidad Pedagógica Nacional Francisco Morazán, Tegucigalpa, Honduras. Email: **danguefl@doctor.upv.es**

†Institut de Matemàtica Multidisciplinar, Universitat Politècnica de València, 46022, València, Spain. Email: jmarinma@imm.upv.es, jmasm@imm.upv.es

closer to the original system. Both requirements make difficult the choice of the appropriate α. It may be necessary to restart the factorization more than once, increasing α on each restart until breakdown is avoided.

We propose a method that obtains a preconditioner for the system (2), updating an IC factorization computed for the regularized LS problem C_α by removing the shift applying a bordering technique.

Consider the matrix

$$(C_\alpha - \beta I) \tag{4}$$

which is an update of the shifted matrix C_α. The closer β is to α, the closer this update is to the normal equations of the original system.

Our technique consists on computing a preconditioner for the normal equations of the original system, updating an incomplete Cholesky factorization obtained for $C_\alpha \approx L_\alpha^T L_\alpha$ using the augmented matrix

$$\begin{bmatrix} C_\alpha & \beta^{1/2} I \\ \beta^{1/2} I & I \end{bmatrix}. \tag{5}$$

Observe that matrices (4) and (5) are related by the linear operators

$$C_\alpha - \beta I = \begin{bmatrix} I & O \end{bmatrix} \begin{bmatrix} C_\alpha & \beta^{1/2} I \\ \beta^{1/2} I & I \end{bmatrix} \begin{bmatrix} I \\ \beta^{1/2} I \end{bmatrix} \tag{6}$$

and their inverses

$$(C_\alpha - \beta I)^{-1} = \begin{bmatrix} I & O \end{bmatrix} \begin{bmatrix} C_\alpha & \beta^{1/2} I \\ \beta^{1/2} I & I \end{bmatrix}^{-1} \begin{bmatrix} I \\ O \end{bmatrix}. \tag{7}$$

Thus an IC factorization of the augmented matrix (5) can be used as a preconditioner for the original normal equations using relations (6) and (7). A similar technique was used successfully in [3] for updating preconditioners for linear systems.

We will present results for matrices from the Florida Sparse Matrix Collection [4] that show that our method is competitive compared with another techniques recently used for solving rank deficient least square problems.

Acknowledgements

This research is supported by the Spanish Ministerio de Economía y Competitividad under grants MTM2014-58159-P and MTM2015-68805-REDT.

References

[1] Björck, Å.: Numerical methods for Least Squares Problems. SIAM, Philadelphia (1996)

[2] Bru, R., Marín, J., Mas, J., Tůma, M.: Preconditioned iterative methods for solving linear least squares problems. SIAM J. Sci. Comput. **36**(4), A2002–A2022 (2014)

[3] Cerdán, J., Marín, J., Mas, J.: Low-rank updates of balanced incomplete factorization preconditioners. Numer. Algorithms **74**, 337–370 (2017).

[4] T. A. Davis, Y. Hu, The university of florida sparse matrix collection, ACM Trans. Math. Softw. 38 (1) (2011) 1:1–1:25.