Local convergence of some third-order iterative methods for multiple roots

Diego Alarcón [a], Fabricio Cevallos [b], Jose L. Hueso [c], Eulalia Martínez. [c]

[a] *Universitat Politècnica de València, Spain*

[b] *Facultad de Ciencias Económicas. Universidad Laica Eloy Alfaro de Manabí, Ecuador*

[c] *Instituto de Matemática Multidisciplinar. Universitat Politècnica de València, Spain*

Abstract

We are interested in finding approximate multiple roots of nonlinear equations by using iterative methods. We consider the nonlinear equation $f(x) = 0$, where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, with a solution α of multiplicity m, that is, verifying $f^{(j)}(\alpha) = 0, \quad j = 0, 1, \cdots, m-1$, and $f^{(m)}(\alpha) \neq 0$. It is a special case where some particular aspects must be taken into account. In this sense different iterative methods for this particular case have been recently published see [1]-[3] and the references therein. The case of multiple roots appears, for example, in the study of the Van der Waals equation of state, in the compression of band-limited signals and the multipactor effect in electronic devices among other phenomenons.

Given an iterative method, We say that r is the radius of the local convergence ball if the sequence x_n generated, starting from any initial point in the open ball $B(\alpha, r)$ converges to α and remains in the ball. In these studies it is interesting to obtain the largest possible value of r, but obviously, this depends on the conditions that the nonlinear function verifies.

Specially interesting from a mathematical point of view are papers [1] and [4] where a complete local convergence study has been performed, obtaining the convergence radius of the well-known modified Newton, Halley and Osada's methods for multiple zeros, when the involved function satisfies a Hölder or a center-Hölder continuity condition, thah is, $\forall x, y \in D$, $p \in]0, 1]$ and K_0, K_m positive real numbers,

$$\left| f^{(m)}(x^*)^{-1}(f^{(m+1)}(x) - f^{(m+1)}(y)) \right| \leq K_0 |x - y|^p,$$
$$\left| f^{(m)}(x^*)^{-1} f^{(m+1)}(x) \right| \leq K_m.$$

For this purpose, different results involving sophisticated properties of divided differences have been used.

In this work we give a simple alternative to obtain this local convergence radius for iterative methods for nonlinear equations with multiple roots. We point out that similar results can be obtained in a much more natural way. In this case we will use the following bound conditions for function f that defines the nonlinear problem:

$$|f^{(m)}(\alpha)^{-1} f^{(m+1)}(x)| \leq k_1, \quad \forall x \in D, \quad k_1 > 0.$$

In case the method uses second derivative we need a second assumption as follows:

$$|f^{(m)}(\alpha)^{-1}f^{(m+2)}(x)| \leq k_2, \quad \forall x \in D, \quad k_2 > 0.$$

First of all We develop the whole study for two third-order iterative methods due to Osada whose iterative expression is given by (1) having similar results that the ones obtained in precedent studies cited before.

$$x_{n+1} = x_n - \frac{1}{2}m(m+1)\frac{f(x_n)}{f'(x_n)} + \frac{1}{2}(m-1)^2\frac{f'(x_n)}{f''(x_n)}. \tag{1}$$

An analogous procedure can be applied in order to obtain the radius of other iterative methods. Specifically We have also studied the third order method introduced by Dong in [5], whose expression is given by:

$$y_n = x_n - \sqrt{m}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - mb\frac{f(y_n)}{f'(x_n)}$$

where $b = \left(1 - \frac{1}{\sqrt{m}}\right)^{(1-m)}$, the method has two steps and because of that some variations have to be taken into account.

Key words: Nonlinear equations, Iterative methods, Multiple roots, ball's local convergence radius.

AMS Subject Classification: 65H05, 65H10.

References

[1] W. Bi, H. Ren, Q. Wu, Convergence of the modified Halley's method for multiple zeros under Hölder continuous derivative, Numer. Algorithms 58 (2011) 497-512.
[2] Xiaojian Zhou, Xin Chen, Yongzhong Song, On the convergence radius of the modified Newton method for multiple roots under the center–Hölder condition, Numer. Algor. 65 (2014) 221-232.
[3] Xiaojian Zhou, Yongzhong Son, Convergence radius of Osada's method under center-Hölder continuous condition, Applied Mathematics and Computation 243 (2014) 809-816.
[4] I. Argyros, On the convergence and application of Newton's method under weak Hölder continuity assumptions, Int. J. Comput. Math. 80 (2003) 767-780.
[5] C. Dong, A basic theorem of constructing an iterative formula of the higher order for computing multiple roots of an equation, Mathematica Numerica Sinica, vol. 11, (1982) 445-450.