The Randomized Non-Autonomous Scaled Logistic Differential Equation: Theory and Applications

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1 Introduction

In this contribution we study the randomization of the classical logistic differential equation

$$X'(t) = K(t)(1 - X(t))X(t), \quad t > t_0 \in \mathbb{R}, \quad X(t_0) = X_0. \qquad (1)$$

We will assume that the initial condition X_0 is an absolutely continuous real random variable (r.v.) and the diffusion coefficient, $K(t)$ is a stochastic process (s.p.). defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the sake of convenience, it is assumed the equation is normalized, that is,

$$\mathbb{P}\left[\{\omega \in \Omega : 0 < X_0(\omega) < 1\}\right] = 1.$$

We compute approximations of the first probability density function (1-p.d.f.), $f_1(x,t)$, of the solution s.p. $X(t)$ to the random non-linear differential with initial condition (1). The computation of the 1-p.d.f. of the solution s.p.

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is advantageous since it permits to compute all one-dimensional statistical moments of the solution s.p.

$$\mathbb{E}\left[(X(t))^k\right] = \int_{-\infty}^{+\infty} (x(t))^k f_1(x,t)\,\mathrm{d}x, \quad k = 0, 1, 2, \ldots$$

As a consequence, the mean $\mu_X(t) = \mathbb{E}[X(t)]$ and the variance, $\sigma_X^2(t) = \mathbb{V}[X(t)] = \mathbb{E}[(X(t))^2] - (\mu_X(t))^2$, are easily derived as particular cases. Furthermore, $f_1(x,t)$ allows us to compute the probability that the solution lies in specific sets of interest,

$$\mathbb{P}\left[\{\omega \in \Omega : a \leq X(t)(\omega) \leq b\}\right], \quad -\infty \leq a < b \leq +\infty.$$

To conduct our study, we will apply to different techniques. First, the so-called Karhunen-Loève expansion (KLE) and, secondly, the Random Variable Transformation (RVT).

The KLE allows us to represent the second-order s.p. $K(t)$, i.e., $\mathbb{E}[K^2(t)] < +\infty$ for all t, as a function of a denumerable set of r.v.'s, say $\{\eta_i : i \geq 1\}$ having zero-mean, unit variance and uncorrelated, $\mathbb{E}[\eta_i] = 0$, $\mathbb{V}[\eta_i] = 1$ and $\mathbb{E}[\eta_i \eta_j] = 0$ for every $i, j \geq 1$, $i \neq j$. KLE constitutes a generalized Fourier-type spectral representation for s.p.'s.

Theorem 1 (KLE expansion) *[3, p.202]. Let us consider the second-order s.p. $\{X(t) : t \in \mathcal{T}\}$ being $\mathcal{T} \subset \mathbb{R}$ and assume that X is a second-order r.v., i.e., $\mathbb{E}[X^2] < +\infty$. Then,*

$$X(t,\omega) = \mu(t) + \sum_{j \geq 1} \sqrt{v_j}\phi_j(t)\eta_j(\omega), \quad \omega \in \Omega, \tag{2}$$

where this sum is mean square convergent and

$$\eta_j(\omega) := \frac{1}{\sqrt{v_j}} \mathbb{E}\left[(X(t) - \mu(t))\phi_j(t)\right],$$

being $\{(v_j, \phi_j(t)) : j \geq 1\}$ the eigenpairs of the covariance $\Gamma_X(t,s)$ operator satisfying the integral equation

$$\lambda_i \phi_n(t) = \int_{\mathcal{T}} \Gamma_X(t,s)\phi_j(s)\,\mathrm{d}s,$$

In its multi-dimensional version, this result can be stated as follows

Theorem 2 (Multidimensional RVT method) *[1, p.25]. Let us consider* $\mathbf{X} = (X_1, \ldots, X_n)^\top$ *and* $\mathbf{Z} = (Z_1, \ldots, Z_n)^\top$ *two n-dimensional absolutely continuous random vectors defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$. *Let* $\mathbf{r} : \mathbb{R}^n \to \mathbb{R}^n$ *be a one-to-one deterministic transformation of* $\mathbf{X}$ *into* $\mathbf{Z}$, *i.e.,* $\mathbf{Z} = \mathbf{r}(\mathbf{X})$. *Assume that* $\mathbf{r}$ *is continuous in* $\mathbf{X}$ *and has continuous partial derivatives with respect to each* X_i, $1 \leq i \leq n$. *Then, if* $f_{\mathbf{X}}(\mathbf{x})$ *denotes the joint probability density function of random vector* $\mathbf{X}$, *and* $\mathbf{s} = \mathbf{r}^{-1} = (s_1(z_1, \ldots, z_n), \ldots, s_n(z_1, \ldots, z_n))^\top$ *represents the inverse mapping of* $\mathbf{r} = (r_1(x_1, \ldots, x_n), \ldots, r_n(x_1, \ldots, x_n))^\top$, *the joint probability density function of random vector* $\mathbf{Z}$ *is given by*

$$f_{\mathbf{Z}}(\mathbf{s}) = f_{\mathbf{X}}\left(\mathbf{h}(\mathbf{z})\right)|J|, \tag{3}$$

where $|J|$, *which is assumed to be different from zero, is the absolute value of the Jacobian defined by the determinant*

$$J = \det\left(\frac{\partial \mathbf{s}^\top}{\partial \mathbf{z}}\right) = \det\begin{pmatrix} \frac{\partial s_1(z_1,\ldots,z_n)}{\partial z_1} & \cdots & \frac{\partial s_n(z_1,\ldots,z_n)}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_1(z_1,\ldots,z_n)}{\partial z_n} & \cdots & \frac{\partial s_n(z_1,\ldots,z_n)}{\partial z_n} \end{pmatrix}. \tag{4}$$

2 General case

I a first step, we consider a general s.p. for the coefficient $K(t)$ and then it is represented via the KLE

We first consider the well-known representation

$$X(t) = Y_0 S_1(t; A) + Y_1 S_2(t; A), \tag{5}$$

$$S_1(t; A) = \sum_{n \geq 0} C_n(A)(t - t_0)^n, \quad S_2(t; A) = \sum_{n \geq 0} D_n(A)(t - t_0)^n, \tag{6}$$

of the solution s.p. of IVP (1)–(**??**) being $S_1(t; A)$ and $S_2(t; A)$ two linearly independent solutions. Then, we take its truncation

$$X_N(t) = Y_0 S_1^N(t; A) + Y_1 S_2^N(t; A), \tag{7}$$

$$S_1^N(t; A) = \sum_{n=0}^{N} C_n(A)(t - t_0)^n, \quad S_2^N(t; A) = \sum_{n=0}^{N} D_n(A)(t - t_0)^n, \tag{8}$$

being N a positive integer previously fixed, to construct the following approximation

$$f_1^N(x,t) = \iint_{\mathcal{D}_{A,Y_1}} f_{A,Y_0,Y_1}\left(z_1, \frac{x - y_1 S_2^N(t;a)}{S_1^N(t;a)}, y_1\right) \times \left|\frac{1}{S_1^N(t;a)}\right| \mathrm{d}y_1\, \mathrm{d}a, \quad (9)$$

to the 1-p.d.f. of the truncated s.p. $X_N(t)$ to IVP (1)–(**??**).

Finally, we will introduce suitable hypotheses to legitimate that

$$\lim_{N\to+\infty} f_1^N(x,t) = f_1(x,t), \quad \text{for each } (x,t) \in \mathbb{R} \times [t_0, +\infty[\text{ fixed}, \quad (10)$$

being

$$f_1(x,t) = \iint_{\mathcal{D}_{A,Y_1}} f_{A,Y_0,Y_1}\left(z_1, \frac{x - y_1 S_2(t;a)}{S_1(t;a)}, y_1\right) \times \left|\frac{1}{S_1(t;a)}\right| \mathrm{d}y_1\, \mathrm{d}a, \quad (11)$$

where $S_1(t,a)$ and $S_2(t;a)$ are defined in (6).

3 An illustrative example

In this example we will display the behaviour of second-order random differential equation

$$X''(t) - AtX(t) = 0, \quad (12)$$

with initial conditions

$$X(0) = Y_0, \quad X'(0) = Y_1. \quad (13)$$

We will consider that A, Y_0 and Y_1 are mutually independent r.v.'s. In a first step, by simplicity it is assumed that each one is uniformly distributed on the interval $]0,1[$, i.e. A, Y_0, $Y_1 \sim U(]0,1[)$. Then, we obtain the 1-p.d.f. of $X_N(t)$, $f_1^N(x,t)$. Figure 1 shows $f_1^N(x,t)$ on the time instant $t = 3$ for different values of N (truncation order). On the left, for $N = 1,\ldots,3$, on the right for $N = 4,\ldots,8$. We can observe that when the truncation increase $f_1^N(x,3)$ converges to the exact 1-p.d.f, $f_1(x,3)$, of the solution s.p. of the IVP (12)–(13).

Figure 1: Plot of $f_1^N(x,3)$ given by (9) for different values of N: $N = 1, \ldots, 3$ (left), $N = 4, \ldots, 8$ (right).

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