Efficient three time level numerical scheme for option pricing Heston-Hull-White model

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1 Introduction

In order to construct a reliable mathematical model for commodities\options have a long time to be expired, it is convenient to consider that its interest rate is controlled by a stochastic process. In Heston model, the underlying asset and its variance are controlled by two stochastic differential equations (SDEs). By adding a third stochastic differential equation for the interest rate, the model is called the Heston-Hull-White (HHW) model. Consider a complete probability space $(\Omega, \mathcal{F}, P)$ with a time domain $[0, T]$, Ω is the set all inquiry of the stochastic economy between 0 and T. $\mathcal{F}$ is the sigma algebra of distinguishable events at time T and P is the risk-neutral probability measure on $\mathcal{F}$. Here the option pricing under the probability measure P is constituted by a system of three stochastic differential equations (SDEs) given by [1, 2]

$$\begin{aligned} dS =& \; rSdt + \sqrt{\nu} S dW_1(t), \\ d\nu =& \; \kappa(\eta - \nu)dt + \sigma_1 \sqrt{\nu} dW_2(t), \\ dr =& \; a(b(t) - r)dt + \sigma_2 dW_3(t), \end{aligned} \tag{1}$$

where S, ν and r are the underlying asset, its variance and interest rate. $W_1(t)$, $W_2(t)$ and $W_3(t)$ are standard Brownian motions associated with

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correlation factors ρ_{12}, ρ_{13} and $\rho_{23} \in (-1,1)$ such that $dW_i dW_j = \rho_{ij} dt$, $1 \leq i,j \leq 3 \wedge j > i$. The speed of the volatility is denoted by $\kappa > 0$, η is the volatility mean, σ_1, and σ_2 are the second order volatility and the volatility of the interest rate respectively. The parameter $a > 0$ determines the speed of mean reversion of the interest rate process and $b(t)$ represents the structure of the interest rate. Based on the non-arbitrage assumptions, the European option under HHW-model is given by [1]-[3]

$$\begin{aligned} &\tfrac{\partial u}{\partial \tau} = \tfrac{1}{2}\nu S^2 \tfrac{\partial^2 u}{\partial S^2} + \tfrac{\sigma_1^2}{2}\nu \tfrac{\partial^2 u}{\partial \nu^2} + \tfrac{\sigma_2^2}{2}\tfrac{\partial^2 u}{\partial r^2} + \rho_{12}\sigma_1 S\nu \tfrac{\partial^2 u}{\partial S \partial \nu} + \rho_{13}\sigma_2 S\sqrt{\nu}\tfrac{\partial^2 u}{\partial S \partial r} \\ &+\rho_{23}\sigma_1\sigma_2\sqrt{\nu}\tfrac{\partial^2 u}{\partial \nu \partial r} + rs\tfrac{\partial u}{\partial S} + \kappa(\eta-\nu)\tfrac{\partial u}{\partial \nu} + a(b(T-\tau)-r)\tfrac{\partial u}{\partial r} - ru, \end{aligned} \tag{2}$$

where $\tau = T - t$, $S, \nu > 0$, $r \in \mathbb{R}$, associated with the boundary conditions:

$$\begin{aligned} &u(0,\nu,r,\tau) = 0, \quad \lim_{S\to\infty}\frac{\partial u}{\partial S} = 1, \quad \frac{\partial u}{\partial r}(S,\nu,r_{\min},\tau) = 0, \\ &\tfrac{\partial u}{\partial r}(S,\nu,r_{\max},\tau) = 0, \quad \lim_{\nu\to\infty} u(S,\nu,r,\tau) = S, \\ &\tfrac{\partial u}{\partial \tau} = \tfrac{\sigma_2^2}{2}\tfrac{\partial^2 u}{\partial r^2} + rS\tfrac{\partial u}{\partial S} + \kappa\eta\tfrac{\partial u}{\partial \nu} + a(b(T-\tau)-r)\tfrac{\partial u}{\partial r} - ru, \quad \nu = 0. \end{aligned} \tag{3}$$

Initial condition for European call option is given by the payoff

$$u(S,\nu,r,0) = \max\{0, S-E\}, \tag{4}$$

where E is the strike price.

2 Problem Transformation

We start this section by applying a suitable mathematical transformation on (2) in order to reduce the number of the mixed derivatives. Consider the new variables (x,y,z) such that

$$x = \sigma_1 \ln S, \quad y = \nu - \rho_{12}\sigma_1 \ln S, \quad z = r - \frac{2\sigma_2(\rho_{23} - \rho_{12}\rho_{13})}{\sigma_1 \tilde{\rho}_{12}^2}\sqrt{\nu}. \tag{5}$$

The new PDE based on the transformation (5) becomes

$$\begin{aligned} &\tfrac{\partial u}{\partial \tau} = \alpha(x,y)\left(\tfrac{\partial^2 u}{\partial x^2} + \tilde{\rho}_{12}\tfrac{\partial^2 u}{\partial y^2}\right) + \hat{\rho}\tfrac{\partial^2 u}{\partial z^2} + \beta(x,y)\tfrac{\partial^2 u}{\partial x \partial z} + \gamma(x,y,z)\tfrac{\partial u}{\partial x} \\ &-\tilde{\gamma}(x,y,z)\tfrac{\partial u}{\partial y} + \hat{\gamma}(x,y,z,\tau)\tfrac{\partial u}{\partial z} - \breve{\gamma}(x,y,z)u, \end{aligned} \tag{6}$$

where

$$
\begin{aligned}
\alpha(x,y) =&\ \tfrac{\sigma_1^2}{2}(y+\rho_{12}x),\\
\hat{\rho} =&\ \tfrac{1}{2}\left(\tfrac{\sigma_2}{\tilde{\rho}_{12}^2}\right)^2((\rho_{12}^2-\tilde{\rho}_{12}^2)\rho_{23}^2-2\rho_{13}\rho_{12}^3\rho_{23}+(\rho_{12}\rho_{13})^2+\tilde{\rho}_{12}^4),\\
\beta(x,y) =&\ \tfrac{\sigma_1\sigma_2(\rho_{13}-\rho_{12}\rho_{23})}{\tilde{\rho}_{12}^2}\sqrt{y+\rho_{12}x},\\
\gamma(x,y,z) =&\ \sigma_1(z+\tfrac{2\sigma_2(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_1\tilde{\rho}_{12}^2}\sqrt{y+\rho_{12}x}-\tfrac{1}{2}(y+\rho_{12}x)),\\
\tilde{\gamma}(x,y,z) =&\ \rho_{12}\gamma(x,y,z)+\kappa((y+\rho_{12}x)-\eta),\\
\hat{\gamma}(x,y,z,\tau) =&\ a(b(T-\tau)-z+\tfrac{2\sigma_2(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_1\tilde{\rho}_{12}^2}\sqrt{y+\rho_{12}x})+\tfrac{\hat{\rho}_1}{\sqrt{y+\rho_{12}x}}-\hat{\rho}_2\sqrt{y+\rho_{12}x},\\
\hat{\rho}_1 =&\ \tfrac{\sigma_2(\rho_{13}\rho_{12}-\rho_{23})(4\kappa\eta-\sigma_1^2)}{4\sigma_1\tilde{\rho}_{12}^2},\\
\hat{\rho}_2 =&\ \tfrac{\kappa\sigma_2}{\tilde{\rho}_{12}^2\sigma_1}(\rho_{12}\rho_{13}-\rho_{23}),\\
\breve{\gamma}(x,y,z) =&\ z+\tfrac{2\sigma_2(\rho_{23}-\rho_{12}\rho_{13})}{\sigma_1\tilde{\rho}_{12}^2}\sqrt{y+\rho_{12}x}.
\end{aligned}
\tag{7}
$$

The boundary conditions and payoff under the transformation are given by

$$
\begin{aligned}
\lim_{x\to-\infty}u(x,y,z,\tau)=0,\quad & \sigma_1\lim_{x\to\infty}\left(\frac{\partial u}{\partial x}-\rho_{12}\frac{\partial u}{\partial y}\right)=e^{\frac{x}{\sigma_1}},\\
\tfrac{\partial u}{\partial z}\Big|_{z_{\min}}=0,\quad & \tfrac{\partial u}{\partial z}\Big|_{z_{\max}}=0,\quad \lim_{y+\rho_{12}x\to\infty}u(x,y,z,\tau)=e^{\frac{x}{\sigma}1},\\
& \tfrac{\partial u}{\partial \tau}=\left(2\sigma_1 z-\tfrac{\kappa\eta}{\rho_{12}}\right)\tfrac{\partial u}{\partial x},\quad y=-\rho_{12}x,\\
& u(x,y,z,0)=\max\{0,e^{\frac{x}{\sigma_1}}-E\}.
\end{aligned}
\tag{8}
$$

Note that under the transformation (5), two mixed derivative are eliminated, consequently, removing eight points stencil from the corresponding finite scheme.

3 The finite difference scheme Construction

The time domain is discretized by the mesh points $\tau_n = n\Delta\tau$, $0 \le n \le N_\tau$, and the spatial variables x, y and z are divided by the nodes $x_i = x_0 + i\Delta x$, $y_j = y_0 + j\Delta y$, $z_k = z_0 + k\Delta z$, $0 \le i \le N_x$, $0 \le j \le N_y$ and $0 \le k \le N_z$. The first partial derivative of the option with respect to time is discretized using the three time-level approximation [4], while first and second order spacial derivatives are discretized using implicit central approximations. Let $u(x_i, y_j, z_k, \tau_n) \approx U_{i,j,k}^n$ consequently, the corresponding finite difference scheme is discretized by

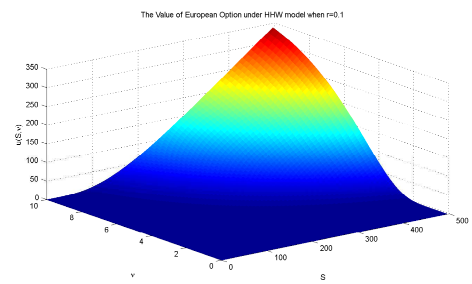
$$\begin{array}{l}\hat{a}_{ijk}(1)U^{n+1}_{i,j,k} + \hat{a}_{ij}(2)U^{n+1}_{i+1,j,k} + \hat{a}_{ij}(3)U^{n+1}_{i-1,j,k} + \hat{a}_{ijk}(4)U^{n+1}_{i,j+1,k} + \hat{a}_{ijk}(5)U^{n+1}_{i,j-1,k} \\ +\hat{a}_{ij}(6)\left(U^{n+1}_{i+1,j,k+1} - U^{n+1}_{i+1,j,k-1} - U^{n+1}_{i-1,j,k+1} + U^{n+1}_{i-1,j,k-1}\right) + \hat{a}^{n}_{ijk}(7)U^{n+1}_{i,j,k+1}, \\ +\hat{a}^{n}_{ijk}(8)U^{n+1}_{i,j+1,k-1} = U^{n}_{i,j,k} - \frac{1}{4}U^{n-1}_{i,j,k},\end{array} \tag{9}$$

where $\hat{a}_{ijk}(l)$, $l = 1, 2 \ldots, 5$, $\hat{a}_{ij}(6)$, $\hat{a}^{n}_{ijk}(7)$ and $\hat{a}^{n}_{ijk}(8)$ are the discretization coefficients. The derivatives at the boundaries are discretized using the central approximations and the points outside the computational domain are eliminated by solving their equations with the scheme (9) at the same points, in order to obtain a finite difference scheme with a truncation error of the second order of the spatial variables and time stepsizes.

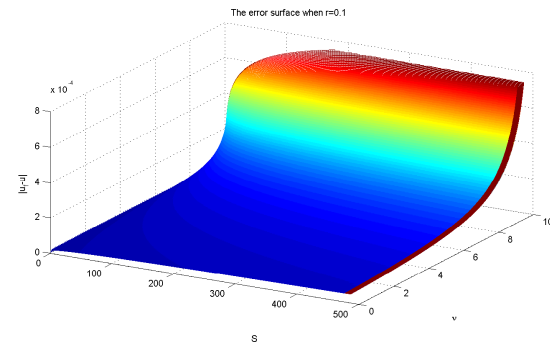
4 Numerical Example

In this section, we provide an example for the associated error of the proposed finite difference scheme (9) using MatLab.

Example 1. Consider an European call option problem (2)-(4) under HHW-model with parameters $T = 1$, $E = 100$, $S_{\max} = 14E$, $\nu_{\max} = 10$, $r_{\min} = -1$, $r_{\max} = 1$, $\sigma_1 = 0.04$, $\sigma_2 = 0.03$, $\kappa = 3$, $\eta = 0.12$, $a = 0.2$, $b(t) = 0.05 - 0.01e^{-t}$, $\rho_{12} = 0.6$, $\rho_{13} = 0.2$, $\rho_{23} = 0.4$. Consequently, $x \in [-3, 3]$, $y \in [-2, 12]$ and $z \in [-1.2, 1]$, the number of mesh points are chosen such that $N_x = 20$, $N_y = 50$, $N_z = 10$ and $N_\tau = 40$. The value of the option is calculated at a finer grid $(100, 200, 100, 100)$ and $(20, 50, 10, 40)$, after that we obtain the absolute difference as a function in (S, ν, r) at the expiration time. Figure (1.a) shows the option value as a function of (S, ν) and Figure (1.b) reveals the associated absolute error.



(1.a) The option price u as a function of S and ν.



(1.b) The Error surface when $r = 0.1$

References

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