New insights on multiplex PageRank

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Abstract.

Since the introduction of the PageRank algorithm - originally devised by the founders of Google [16]- to sort web pages, much research has been done in order to improve both the numerical method and the range of applications. In this respect, in the latest research papers one can find new numerical methods for computing PageRank (see, e.g., [20], [21], [19], [22], [15], [9]) and a myriad of new applications (see, e.g., [1], [18], [12], [14], and the dedicated article [7]) including those applications related to the emerging topic of multiplex networks, like the studies in [2] and [5]. It is also worth noting the technique of extension of PageRank by using higher-order Markov chains (that is, chains that depend on previous states of the surfer), see [8] and the references therein.

We are interested in a particular feature of the PageRank algorithm: its capability of biasing the PageRank -and therefore the resulting ranking- to some particular nodes. This biasing is done by means of the so-called personalization vector, see [13], [3]. Given that the PageRank vector is the dominant unitary positive eigenvector of a stochastic matrix, and by using basic matrix algebra, it is easy to explicitly write the PageRank vector π, associated to a network of n nodes, as the product of a personalization vector v times a nonsingular matrix X. That is, in the form

$$\pi^T = v^T X \tag{1}$$

where X is a nonsingular $n \times n$ matrix, and $\pi, v \in \mathbb{R}^{n\times 1}$.

By using (1) and some properties of matrix X, one can obtain a useful result: giving a network, the value of the PageRank of each node can only attain values inside a precise subinterval of $(0, 1)$ depending on the entries of matrix X. In more detail, the bounds are defined by the minimum of each column of X and the values of the diagonal entries of X (in [6] it is shown precisely how this localization of the PageRank takes place). Consequently, we have that the biasing produced by the personalization vector v is limited.

In this talk, we present a result about the localization of a particular class of PageRank. We refer to this PageRank as the multiplex PageRank, that is, an ad-hoc PageRank that

has been introduced to deal with the problem of multiplex graphs (graphs composed by several layers with the same number of nodes).

There are different ways to define multiplex PageRank (see, for example [10] and [4]), but we use the one introduced in [17]. According to this approach the multiplex PageRank is the unitary positive eigenvector of a stochastic matrix M_k associated to the multiplex equipped with k layers with n nodes on each layer. Matrix M_k is of the form

$$M_k = \frac{1}{k}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathbb{R}^{2kn\times 2kn}, \tag{2}$$

where B_{11} gathers the information about the topology of the network, B_{22} takes into account the personalization vectors and B_{21} and B_{21} are diagonal matrices.

In this new formulation, personalization vectors are also considered. In fact, we consider one personalization vector for each layer. In the same manner as in the classic PageRank, the multiplex PageRank changes when there exists a change in the personalization vectors. Once again, this change is delimited and its magnitude can be precisely described. As a matter of fact, this is the goal of our communication: to find precise bounds for the multiplex PageRank when different values of the personalization vectors are taken into account.

In the talk we will also show some numerical examples and comment about new lines of research on this field.

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